Plan

I. Anti-self-dual connections, Atiyah-Hitchin-Singer complex, Kuranishi model

II. Generic smoothness, orientations

III. Uhlenbeck compactness, Taubes gluing

IV. Donaldson's theorem, Polynomial invariants

V. Donaldson-Thomas invariants,
G_2, Spin(7)-instantons

VI. Bubbling-off in higher dimensions

VII. Gluing construction of Spin(7)-instantons
by Walpuski

VIII. ―――――― G_2-instantons
by Walpuski

Gauge theory in higher dimensions

Donaldson–Thomas

(I) "complexification" of real 3, 4-dimensional gauge theories:

- 3-dimensional Casson invariants

$\Longrightarrow$ Donaldson–Thomas invariants on a Calabi–Yau 3-fold.

(Thomas, Pandharipande–Thomas, Joyce–Song, Kontsevich–Soibelman, et al.)

- 4-dimensional Donaldson invariants

$\Longrightarrow$ D-T4 invariants on a Calabi–Yau 4-fold.

(Borisov–Joyce, Cao–Leung, et al.)

(II) Gauge theories on manifolds with special holonomy

$$\Longrightarrow \begin{cases} G_2\text{- instantons}, \\ \mathrm{Spin}(7)\text{-instantons}. \end{cases}$$

Hitchin–Kobayashi correspondence

X: compact Kähler surface

$E \to X$, hermitian vector bundle on X

$$\left\{ \begin{array}{l} \text{irreducible} \\ \text{ASD connections} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{"stable" holomorphic} \\ \text{vector bundles} \end{array} \right\}$$

- rather well-understood objects in Algebraic Geometry.
- also useful to calculate the polynomial invariants.

More generally, this holds for Hermitian-Yang-Mills connections on a compact Kähler manifold of $\dim_{\mathbb{C}} = n \geq 1$.

(Donaldson, Uhlenbeck-Yau, et al.)

Donaldson-Thomas invariants

X : Calabi-Yau 3-fold

want to count semistable sheaves on X.

Two issues:

(A) obstructedness of the moduli space of semistable sheaves on X.

(B) compactness of ____.

Thomas sorted out these by using Algebraic Geometry.

(B) $\Longrightarrow$ can use Gieseker compactification, it is kind of automatic in the algebro-geometric setting.

Ⓐ $\Rightarrow$ can construct (symmetric) perfect obstruction theory:

Automorphisms: $\mathrm{Ext}^0(E,E) \cong \mathbb{C}$, if semistable = stable.

tangent space: $\mathrm{Ext}^1(E,E)$,

obstruction space: $\mathrm{Ext}^2(E,E)$,

they are dual to each other via Serre duality and $K_X \cong \mathcal{O}_X$ (called "symmetric")

higher obstruction space: $\mathrm{Ext}^3(E,E)$

$\cong \mathrm{Ext}^0(E,E)$

Serre duality + $K_X \cong \mathcal{O}_X$

$\cong \mathbb{C}$, if semistable = stable.

So, you are in the situation of having an Atiyah-Hitchin-Singer like complex.

$\Rightarrow$ A "perfect obstruction theory" by Behrend-Fantechi (developed in GW-theory) produces a class ("virtual fundamental class) in the Chow group of the moduli space M in the degree being the expected dimensions: $[M]^{\mathrm{vir}} \in A_*(M)$, if semistable = stable.

Donaldson-Thomas invariants

The index (= − the virtual dimensions) is always zero.

⇒ The invariants are just counting, i.e., they are defined as

$$\int_{[M]^{vir}} 1.$$

More generally, Joyce-Song defines the invariants for the case semistable ≠ stable.

MNOP Conjecture

relating rank 1 Donaldson-Thomas invariants with GW invariants.

(Maulik-Nekrasov-Okounkov-Pandharipande)

G_2-instantons

$$\begin{cases} (X, \phi, g) : G_2\text{-manifold} \\ P : \text{principal } G\text{-bundle over } X \\ G : \text{compact Lie group} \end{cases}$$

$\Rightarrow$ $\Omega^2(\mathfrak{g}_P)$ decomposes into

$$\Omega^2(\mathfrak{g}_P) = \Omega^2_7(\mathfrak{g}_P) \oplus \Omega^2_{14}(\mathfrak{g}_P).$$

Def. A : connection on P is a G_2-instanton, if

$$\pi^2_7(F_A) = 0, \qquad \cdots (*)$$

where $\pi^2_7 : \Omega^2(\mathfrak{g}_P) \to \Omega^2_7(\mathfrak{g}_P)$ is the projection.

Rmk : Since

$$\Omega^2_{14}(\mathfrak{g}_P) = \{\alpha \in \Omega^2(\mathfrak{g}_P) \mid *_g(\alpha \wedge \phi) = -\alpha\},$$

$(*)$ can be written as

$$*_g(F_A \wedge \phi) = -F_A.$$

It is also convenient to write the G_2-instanton equation in the following form:

$$F_A \wedge \psi = 0,$$

where $\psi = *_\phi \phi \in \Omega^4(X)$, or

$$*(F_A \wedge \psi) + d_A \xi = 0,$$

where $\xi \in \Omega^0(\mathfrak{g}_P)$. Note that $\xi \equiv 0$ if A is irreducible.

We then consider the following system of the equations:

$$\begin{cases} *(F_A \wedge \psi) + d_A \xi = 0, \\ d_{A_0}^* a = 0, \end{cases}$$

where $A = A_0 + a$ with $a \in \Omega^1(\mathfrak{g}_P)$.

<u>Linearisation</u>

$$L_A : \Omega^0(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P) \longrightarrow \Omega^0(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P), \text{ where}$$

$$L_A = \begin{pmatrix} 0 & d_A^* \\ d_A & *(\psi \wedge d_A) \end{pmatrix}.$$

Spin(7)-instantons

(X, Ω, g) : Spin(7)-manifold

$P \to X$: principal G-bundle over X

G : compact Lie group

$\Rightarrow$ $\Omega^2(\mathfrak{g}_P)$ decomposes into

$$\Omega^2(\mathfrak{g}_P) = \Omega^2_7(\mathfrak{g}_P) \oplus \Omega^2_{21}(\mathfrak{g}_P).$$

Def A: connection on P is a Spin(7)-instanton, if $\pi^2_7(F_A) = 0$, where $\pi^2_7 : \Omega^2(\mathfrak{g}_P) \to \Omega^2_7(\mathfrak{g}_P)$ is the projection.

The deformation complex is given by

$$0 \to \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^7} \Omega^2_7(\mathfrak{g}_P) \to 0,$$

where $d_A^7 := \pi^2_7 \circ d_A$.

Rmk As in the G_2-instanton case, the above Spin(7)-instanton equation can be written as

$$*_g(F_A \wedge \Omega) = -F_A.$$

D-T 4 invariants (Borisov-Joyce, Cao-Leung)

$$\begin{cases} X: \text{Calabi-Yau 4-fold} \\ \theta_0: \text{holomorphic 4-form on } X \end{cases}$$

Complex Hodge star operator

$$*_{\theta_0}: \Omega^{0,2}(X) \longrightarrow \Omega^{0,2}(X)$$

defined as $\quad a \wedge *_{\theta_0} b = (a, b)\, \overline{\theta_0}$

for $a, b \in \Omega^{0,2}(X)$.

$$\Rightarrow \Omega^{0,2}(X) \cong \Omega^{0,2}_{+} \oplus \Omega^{0,2}_{-}.$$

Complex ASD equations

$$(1 + *_{\theta_0}) F_A^{0,2} = 0,$$

$$F_A^{1,1}, \omega = 0, \quad \omega: \text{Kähler form on } X.$$

These form an elliptic system with a gauge fixing.
(In fact, the equations are equivalent to the Spin(7)-instanton equations on a Calabi-Yau 4-fold.)

- Borisov-Joyce constructed a deformation invariant out of the moduli space of solutions to the above equations on a Calabi-Yau 4-fold.

((-2)-shifted symplectic structure $\Rightarrow$ d-manifold structure.)

Constructions of G_2-instantons

- Walpuski '13 on Joyce's compact G_2-manifolds
- ———— '13 adiabatic construction
- Sá Earp-Walpuski '13 :

  construction on

  Kovalev / Corti-Haskins-Nordström-Paccini's

  compact G_2-manifolds

  — Examples :

  - Walpuski '15
  - Menet-Nordström-Sá Earp '15
- Clarke '14 :

  on Bryant-Salamon's G_2-manifolds

  with $G = SU(2)$

Construction of Spin(7)-instantons

- Lewis '98 : on Joyce's first examples of compact Spin(7)-manifolds
- T '12 : on Joyce's second examples of compact Spin(7)-manifolds
- Walpuski '13 : adiabatic construction
- Clarke '13 : on Bryant–Salamon's Spin(7)-manifolds with $G = SU(2)$

cf. Studies on Yang–Mills connections in high dimensions from physicists' perspective: Corrigan–Devchand–Fairlie–Nuyts '83, Ward '84, Fairlie–Nuyts '84, Fubini–Nicolai '85, Acharya–O'Loughlin–Spence '97, Baulieu–Kanno–Singer '98, Kanno–Yasui '00 and other people.

Bubbling-off in higher dimensions

X : compact n-dimensional smooth manifold

$P \to X$, principal G-bundle over X

G : compact Lie group

Th. (compactness theorem, Nakajima '88)

$\{A_i\}_{i \in \mathbb{N}}$: a sequence of Yang-Mills connections

with $\int_X |F_A|^2 \, dV \leq C < \infty$.

Hausdorff measure

$\Rightarrow \exists \Lambda \subset \mathbb{N}$,

$\exists \underset{\text{closed}}{S \subset X}$ with $\mathcal{H}^{n-4}(S) < \infty$,

$\exists Q$: a principal G-bundle over $X \setminus S$,

$\exists A$: a Yang-Mills connection on Q

such that

$\forall \underset{\text{compact}}{K \subset} X \setminus S$, $\exists g_j^K : P|_K \to Q|_K$, $j \in \Lambda$

so that

$\{g_j^K(A_j)\}_{j \in \Lambda}$ converges to A

in C^∞-topology on K.

Th. (Removal singularity theorem, Nakajima '87)

A: smooth Yang-Mills connection on G-vector bundle E over $B_1(0) \setminus \{0\}$.

$\Rightarrow \exists \varepsilon > 0$ such that if $\int_{B_1(0)} |F_A|^2 dV < \varepsilon$

then $\exists$ gauge transformation g such that

$g^*(E)$ extends to a smooth G-vector bundle

$\hat{E}$ over $B_1(0)$ and g^*A extends to a smooth Yang-Mills connection $\hat{A}$ on $\hat{E}$.

Ω-ASD connections

X: compact oriented smooth n-manifold

g: Riemannian metric on X

$*_g$: the Hodge star operator

Ω: closed $(n-4)$-form on X

$E \to X$: unitary vector bundle over X

Def. A: connection on E is an Ω-ASD,

if $*_g F_A = -F_A \wedge \Omega$.

Rmk Ω-ASDs are Yang-Mills connections.

$\because$) $d_A^* F_A = -*_g d_A *_g F_A = -*_g d_A(-F_A \wedge \Omega)$

$= 0$, since $d_A F_A = 0$ (Bianchi identity)

and Ω is closed. $\square$

Examples

- G_2-instantons : Ω = parallel closed 3-form

- Spin(7)-instantons : Ω = parallel closed 4-form

- Hermitian-Yang-Mills connections :

$$\Omega = \frac{\omega^{m-2}}{(m-2)!} \qquad (n = 2m)$$

- Complex ASD : $\Omega = 4\,\mathrm{Re}(\theta_0) + \frac{1}{2}\omega^2$.

θ_0 : holomorphic 4-form, ω : Kähler form.

Th. (Tian)

$\{A_i\}_{i\in\mathbb{N}}$: sequence of Ω-ASD connections.

$\Rightarrow$ ${}^\exists \Lambda \subset \mathbb{N}$,

${}^\exists S$: $(m-4)$-rectifiable subset in X,

${}^\exists \{g_i\}_{i\in\Lambda}$: gauge transformations over $X\setminus S$,

${}^\exists A$: Ω-ASD over $X\setminus S$

such that

$\{g_i(A_i)\}_{i\in\Lambda}$ converges to A on $\forall K \underset{\text{compact}}{\subset} X\setminus S$.

In addition, ${}^\exists \Theta : S \to \mathbb{Z}$, integral density, such that $\lim_{i\to\infty} c_2(A_i) = c_2(A) + c_2(S, \Theta)$,

where $c_2(S,\Theta)(\varphi) := \frac{1}{8\pi^2}\int_S (\varphi, \Omega|_S)\,\Theta\, d\mathcal{H}^{m-4} \llcorner S$

for $\varphi \in \Omega^{m-4}(X)$.

Moreover, if Ω is a calibration, then (S, Θ) is an Ω-calibrated cycle.

rectifiable set

$E \subset \mathbb{R}^n$: <u>m-rectifiable</u>,

if $\exists \{f_i\}$: a countable collection of continuously differentiable maps $f_i : \mathbb{R}^m \to \mathbb{R}^n$

such that $\mathcal{H}^m \left(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^m) \right) = 0$.

Hausdorff measure

Sketch of proof

Th. (Monotonicity formula, Price '83, Tian '00)

For any $0 < \sigma < \rho < r_p$, r_p: injective radius at $p \in X$,

$$\rho^{4-n} e^{a\rho^2} \int_{B_\rho(p)} |F_A|^2 dV_g$$

$$- \sigma^{4-n} e^{a\sigma^2} \int_{B_\sigma(p)} |F_A|^2 dV_g$$

$$\geq 4 \int_{B_\rho(p) \setminus B_\sigma(p)} r^{4-n} e^{ar^2} \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 dV_g,$$

where a is a constant.

Th. (Curvature estimate, Uhlenbeck '82, Tian '00)

A : Yang-Mills connection

$\Longrightarrow$ $\exists\, \varepsilon > 0$, $\exists\, C = C(n) > 0$ such that

for $\forall p \in X$ and $\rho < r_p$,

if $\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 \, dV_g < \varepsilon$,

then $|F_A|(p) \leq \dfrac{C}{\rho^2} \left(\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 \, dV_g \right)^{\frac{1}{2}}$.

Blow-up locus

$$S_b(\{A_i\}) := \bigcap_{r>0} \left\{ x \in X \;\middle|\; \liminf_{i\to\infty} e^{ar^2} r^{4-n} \int_{B_r(x)} |F_A|^2 \, dV_g < \varepsilon \right\}.$$

$\underset{\text{monotonicity formula}}{\Longrightarrow}$ $\mathcal{H}^{n-4}(S_b(\{A_i\})) < \infty$.

$\underset{\text{Curvature estimate}}{\Longrightarrow}$ convergence outside $S_b(\{A_i\})$.

Normal direction

Th. (Tian)

$\{A_i\}$: a sequence of Ω-ASD connections.

Assume that $x \in S$ satisfies

(1) $\exists !$ tangent plane $V = T_x S \subset T_x X$,

(2) $\textcircled{H}(x) \geq \varepsilon_0 > 0$ and

$$\lim_{r \to 0} r^{4-m} \int_{B_r(x)} |F_A|^2 \, dV_g = 0.$$

Then, $\exists\, \sigma : T_x X \to T_x X$, linear transformation

such that a subsequence of $\{\sigma_i^* \exp_x^* A_i\}$

converges to an Ω_x-ASD connection B on $T_x X$

with $F_B \neq 0$, where $\Omega_x := \Omega|_{T_x X}$.

Moreover, $v \lrcorner F_B \equiv 0$ for any $v \in V$.

More detail on $\mathcal{H}^{n-4}(S_h(\{A_i\}))<\infty$

Set $E_{i,r} := \{x\in X \mid e^{ar^2} r^{4-n}\int_{B_r(x)} |F_{A_i}|^2 dV_g \geq \varepsilon\}$.

$\underset{\text{monotonicity}}{\Longrightarrow}$ $E_{i,r} \subset E_{i,r'}$ for $r\leq r'$.

$\underset{\text{diagonal argument}}{\Longrightarrow}$ $\exists \{i_j\}$: a subsequence of $\{i\}$ such that for each k $E_{i_j, 2^{-k}}$ converges to $E_{2^{-k}}$.

Then $E_{2^{-k}} \subset E_{2^{-l}}$ for $k\geq l$.

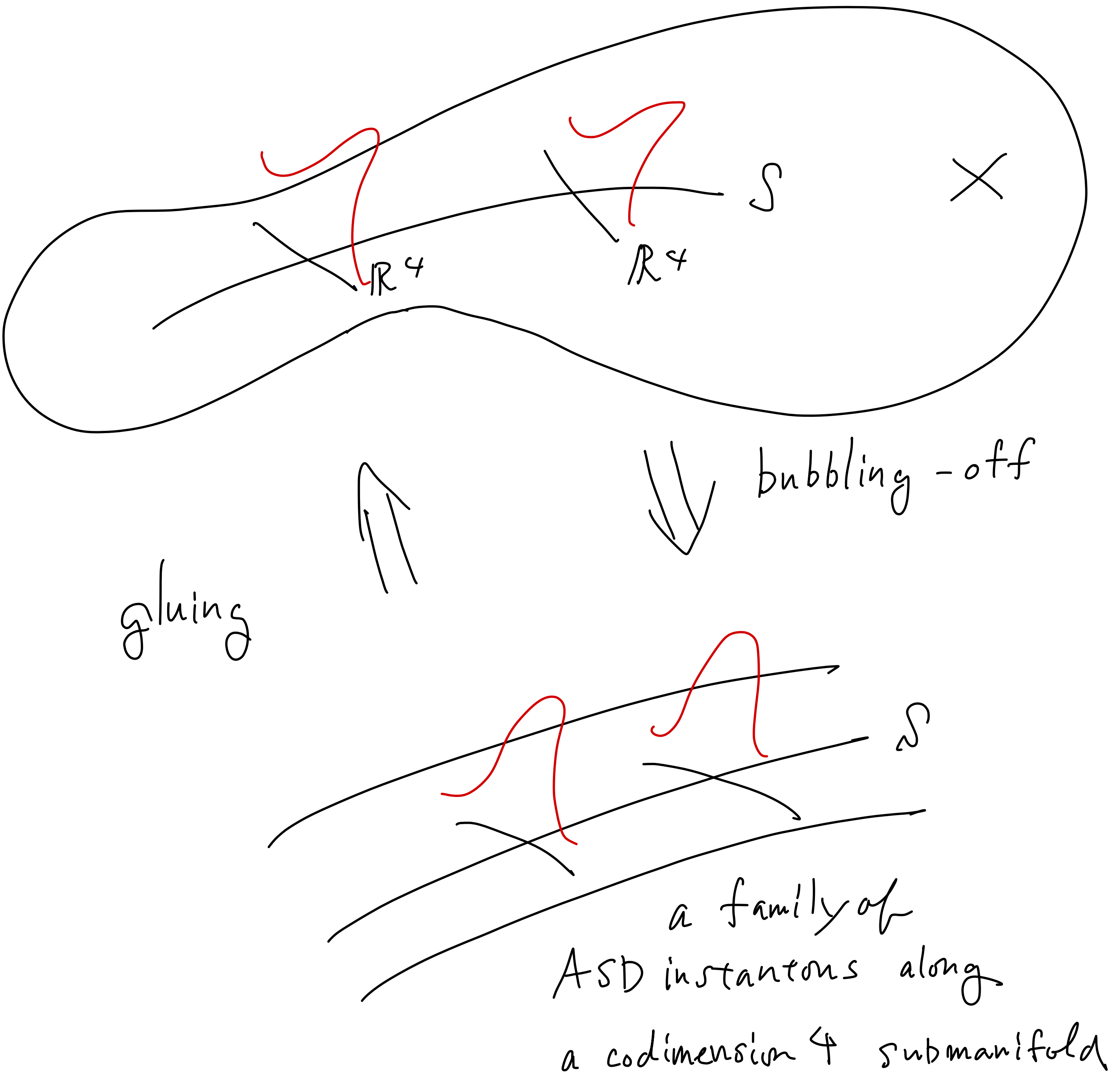
So, set $S := \bigcap_k E_{2^{-k}}$.

$\underset{\text{again monotonicity}}{\Longrightarrow}$ $\mathcal{H}^{n-4}(S)<\infty$.

(note that $S_h(\{A_i\}) \subset S$.)

Gluing construction of Spin(7)-instantons

by Walpuski



Let:

(X, Ω, g): Spin(7)-manifold

Q: Cayley submanifold

Instanton moduli bundle

$$\begin{cases} M : \text{the moduli space of framed ASD instantons} \\ \qquad \text{on a principal } G\text{-bundle over } \mathbb{R}^4 \\ E_\infty : \text{principal } G\text{-bundle over } Q \quad \left(G : \text{compact Lie group} \right) \\ A_\infty : \text{connection on } E_\infty \end{cases}$$

Define

$$\mathcal{M} := \left(Fr(NQ) \times E_\infty \right) \times_{SO(4) \times G} M$$
$$\downarrow$$
$$Q.$$

Then,

a section of $\mathcal{M}$ $\Longleftrightarrow$ a family of ASD instantons on Q.

However, all the sections would not be glued together to a Spin(7)-instanton on X, i.e., we need a control on the horizontal direction to do that.

Fueter operator

Recall the identification $\Lambda^+ T^*Q \cong \Lambda^+ N^*Q$.

Take a local orthonormal frame $\langle I_i \rangle$ of $so(TQ) \cong so(NQ)$.

Define

- $\mathrm{Hom}_{\mathbb{H}}(TQ, NQ)$
$$:= \{ L \in \mathrm{Hom}(TQ, NQ) \mid \sum_i I_i L I_i = -3L \}$$
$$\subset \mathrm{Hom}(TQ, NQ),$$
- $\gamma L := L - \sum_i I_i L I_i$, $L \in \mathrm{Hom}(TQ, NQ)$.

This gives the projection $\mathrm{Hom}(TQ, NQ) \to \mathrm{Hom}_{\mathbb{H}}(TQ, NQ)$.

<u>Def</u> $F_Q : \Gamma(Q, NQ) \to \Gamma(Q, \mathrm{Hom}_{\mathbb{H}}(TQ, NQ))$,
$$F_Q(n) := \gamma(\nabla n).$$
We call this Fueter operator.

<u>Def</u> A Cayley submanifold Q is <u>unobstructed</u>, if F_Q is surjective.

$$\mathcal{V}\mathcal{M} := \mathrm{Fr}(NQ \times E_\infty) \times_{SO(4)\times\mathcal{G}} TM$$
$$\downarrow$$
$$Q.$$

(vertical tangent bundle)

Take $s \in \Gamma(\mathcal{M})$ and consider

- $\mathrm{Hom}_\Omega(TQ, s^*\mathcal{V}\mathcal{M}) \subset \mathrm{Hom}(TQ, s^*\mathcal{V}\mathcal{M})$,
- $\gamma : \mathrm{Hom}(TQ, s^*\mathcal{V}\mathcal{M}) \to \mathrm{Hom}_\Omega(TQ, s^*\mathcal{V}\mathcal{M})$.

Define

$$\begin{array}{rcl} \mathcal{F}_\Omega : \Gamma(\mathcal{M}) & \longrightarrow & \Gamma(\mathrm{Hom}_\Phi(TQ, s^*\mathcal{V}\mathcal{M})) \\ \Psi & & \Psi \\ s & \longmapsto & \gamma(\mathbb{D}s), \end{array}$$

where $\mathbb{D} : s \mapsto \mathbb{D}s \in \Omega^1(s^*\mathcal{V}\mathcal{M})$

is defined via connections on N, Q, E_∞.

<u>Def</u> $s \in \Gamma(\mathfrak{M})$ is a Fueter section,

if $\mathfrak{F}_{n}(s) = 0$.

<u>linearised Fueter operator</u>

$$\begin{array}{cccc} F_{s,n} : & \Gamma(s^*V\mathfrak{M}) & \to & \Gamma(\mathrm{Hom}_n(TQ, s^*V\mathfrak{M})) \\ & \cup & & \cup \\ & \tilde{s} & \longmapsto & \gamma(\nabla \tilde{s}). \end{array}$$

<u>Def</u> A Fueter section s is <u>unobstructed</u>,

if the linearised Fueter operator $F_{s,n}$

is surjective.

Th.1 (Walpuski)

(X, Ω) : compact Spin(7)-manifold.

Suppose

- A_0 : irreducible unobstructed Spin(7)-instantons on a G-bundle E_0 over X,
- Q : unobstructed Cayley submanifold,
- s : unobstructed Fueter section of the instanton moduli bundle $\mathfrak{M} \to Q$ associated to $E_0|_Q$.

$\Rightarrow \exists \Lambda > 0$, E : G-bundle with a family of irreducible and unobstructed Spin(7)-instantons $(A_\lambda)_{\lambda \in (0, \Lambda]}$.

Moreover, $A_\lambda \to A_0$ on $X \setminus Q$ as $\lambda \to 0$, and for $\forall x \in Q$ an ASD connection in the equivalence class given by $s(x)$ bubbles off transversely.

Th. 2 (Walpuski)

Suppose

- Q : Cayley submanifold with self-intersection number zero, diffeomorphic to a K3 surface with metric sufficiently close to a hyperkähler metric,
- the induced connection on $\mathcal{N}Q$ is almost flat.

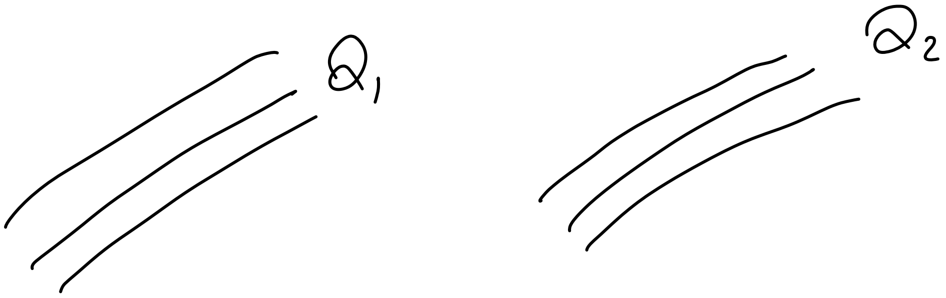
$\Longrightarrow$ There exists a 5-dimensional family of $Spin(7)$-instantons on an $SU(2)$-bundle E over X with $c_2(E) = P.D.[Q]$.

Moreover, if $Q_1, \dots, Q_k$: a collection of k-distinct Cayley submanifolds as above, then there exists an $(8k-3)$-dimensional family of $Spin(7)$-instantons on an $SU(2)$-bundle E over X with $c_2(E) = \sum_{i=1}^{k} P.D.[Q_i]$.

Example

Joyce: X : Spin(7)-manifold
with two disjoint Cayley submanifolds
Q_1, Q_2 of the kind as above.

By Th. 2, Walpuski obtains examples with
$c_2(E) = n\, P.D.[Q_1] + m\, P.D.[Q_2]$
for $\forall n, m \in \mathbb{N}$ by taking small
perturbations of Q_1 and Q_2 as $Q_3, \cdots$.



This includes Lewis' example.

Sketch of proof of Th. 2

- Q: Spin

$\Longrightarrow$ F_Q is the twisted Dirac operator

$$\Gamma(\mathrm{Re}(S_Q^+ \otimes V)) \to \Gamma(\mathrm{Re}(S_Q^- \otimes V)),$$

where $V = S^-_{\nu Q}$.

<u>Fact</u> $\Gamma(\mathrm{Re}(S_Q^+)) \to \Gamma(\mathrm{Re}(S_Q^-))$

is surjective on $K3$.

- the metric on Q is close to a hyperkähler metric and the connection on NQ is almost flat.

$\Longrightarrow$ F_Q is also surjective.

Take

$$\begin{cases} \cdot\ \theta_0 : \text{the trivial connection on the trivial } SU(2)\text{-bundle (index} = -3), \\ \cdot\ \mathcal{M} : \text{the moduli space of rank} = 2 \text{ framed ASD instantons on } \mathbb{R}^4 \text{ with } c_2(E) = 1. \end{cases}$$

Under the assumption, $\mathcal{F}_n$ is surjective, and it has an 8-dimensional kernel.

$\Longrightarrow$ by Th. 1: obtain a 5-dimensional family of Spin(7)-instantons.

Gluing construction of G_2-instantons by Walpuski

$$\begin{cases} (X,\phi,g) : G_2\text{-manifold} \\ P \subset X \quad : \text{associative submanifold} \end{cases}$$

Recall
$$\begin{array}{ccc} TP & \cong & \Lambda^+ N^* P \\ \Psi & & \Psi \\ v & \longmapsto & -i(v)\phi . \end{array}$$

Thinking of $\Lambda^+ N^* P \subset so(P)$, one obtains
$$\gamma : TP \to End(NP).$$

<u>Fueter operator</u>

$$\begin{array}{cccc} F_P : & \Gamma(NP) & \to & \Gamma(NP) \\ & \Psi & & \Psi \\ & v & \longmapsto & \gamma(\nabla v), \end{array}$$

where ∇ is a connection on NP induced by that on X.

Define $\mathcal{F}_\phi : \Gamma(M) \to \Gamma(S^* M)$

in a similar way to the Spin(7)-instanton case.

Linearised Fueter operator

$$F_{s,\phi} : \Gamma(s^*VM) \to \Gamma(s^*VM).$$

This is elliptic, self-adjoint. But it cannot be invertible, if s is a Fueter section, since it comes with a 1-parameter family of Fueter sections $(s_\lambda)_{\lambda \in \mathbb{R}^+}$.

So, we consider $(\phi_t)_{t \in (-T,T)}$: a family of torsion-free G_2-structures on X.

We then have $(P_t)_{t \in (-T',T')}$: a family of associated submanifolds in X, where $T' \in (0,T]$, and $(\mathfrak{M}_t)_{t \in (-T',T')}$: a family of instanton moduli bundles with a family of Fueter operators $(\mathcal{F}_{P_t})_{t \in (-T',T')}$.

Suppose:

s_0: Fueter section of $\mathfrak{M}_0$
with $\dim \operatorname{Ker} F_{s_0,\phi_0} = 1$.

$\Rightarrow$
implicit
function
theorem

$\exists\ (s_t)_{t\in(-T',T')}$: a family of
sections of $\mathfrak{M}_t$ with

$$\mathfrak{F}_t(s_t) + \mu(t)\,\hat{v}\circ s_t = 0,$$

where $\mu : (-T',T') \to \mathbb{R}$, a smooth function vanishing at zero, and $\hat{v} \in \Gamma(v\mathfrak{M})$ is a vector field generated by the action of $\mathbb{R}^+$ on $\mathfrak{M}$.

Def

s_0 is unobstructed with respect to ϕ_0,

if $\left.\dfrac{\partial \mu}{\partial t}\right|_{t=0} \neq 0$.

Th. (Walpuski)

$(\phi_t)_{t\in(-T,T)}$: family of torsion-free G_2-structures.

Suppose

- B : unobstructed G_2-instanton on a G-bundle E_0 over (X, ϕ_0),
- P : unobstructed associative submanifold in (X, ϕ_0),
- S : Fueter section of the instanton moduli bundle $\mathfrak{M}$ over P associated to $E_0|_P$, that is unobstructed with respect to (ϕ_t).

$\Rightarrow$ $\exists \Lambda > 0$, $\exists E$: G-bundle with a family of connections $(A_\lambda)_{\lambda\in(0,\Lambda]}$,

$\exists$ continuous function $t : [0, \Lambda] \to (-T, T)$ with $t(0) = 0$ such that

A_λ : G_2-instanton on E over $(X, \phi_{t(\lambda)})$ for $\lambda \in (0, \Lambda]$.

Moreover, A_λ converges to B on $X \setminus P$, and for $\forall x \in P$ an ASD instanton in the equivalence class given by $s(x)$ bubbles off transversely as $\lambda \to 0$.